Dynamics of incompressible fluid membranes

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A dynamical model for fluid membranes is presented that takes into account local conservation of the membrane area. It is purely relaxational, obeys detailed balance, and is manifestly covariant. The tangential flow due to the conservation of area is described by a velocity field. As an application the dynamical behavior of a nearly cylindrical membrane is considered, including relaxation rates and stability properties. Also, the propagation of perturbations of a planar membrane is investigated, leading to logarithmic corrections to known results for a model without area conservation.

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I. THE MODEL

Fluid membranes are two-dimensional macroscopic systems formed by amphiphilic molecules which had no rigid correlation, but diffuse freely in the membrane area (for a survey, see [1,2]). The diffusive motion is much faster than the geometrical motion of the membrane. Moreover, the membrane has a practically vanishing surface tension.

In the well-known static model by Helfrich [3], the membrane is described by a mathematical, two-dimensional manifold embedded in a three-dimensional space. Its thickness and microscopic structure are not taken into consideration. Consequently, the energy per area is only a function of the local curvature.

We want to extend this theory to a dynamical model which respects the incompressibility of the membrane material [4]. Since there is no exchange of molecules between the membrane and its environment, the membrane area is locally conserved. Thus a deformation of the film is connected with a flow of material. Because of the high in-plane viscosity and of the viscosity of the surrounding fluid, the motion of a membrane is overdamped, and we will choose a purely relaxational model. We will not include hydrodynamics interactions with the surrounding fluid. This is a strong simplification, but makes the mathematical treatment much more simple and makes the effect of incompressibility of the membrane transparent. This model can therefore be seen as a step toward a full treatment of the dynamics of fluid membranes. However, the embedding fluid is incorporated insofar as we have added a pressure difference to our model. On the other hand, a membrane embedded in a gas is perceivable. Dynamical models for fluid membranes embedded in a fluid already exist in the literature, but they are bound to a special geometry [5,6]. In this paper, we consider hydrodynamics on a general curved manifold (the membrane) which itself is moving in time. To describe this situation we have to use the language of (classical) differential geometry.

The time-dependent D-dimensional manifold in a $(D+1)$-dimensional Euclidean embedding space is described by a Euclidean vector $\mathbf{X}(x;t)$, with $(D+1)$ components, each depending on D internal coordinates x^i and time t.

Because of the in-plane diffusion of the molecules there is, contrary to the case of a tethered membrane, no preferred coordinate system, i.e., the theory has to be *covariant* in the language of differential geometry. This means that the equation of motion must be invariant under the change of coordinates

$$x \to x' = x'(x;t) \ . \tag{1}$$

The coordinate transformations are allowed to be time dependent [7]. In order to formulate the model we temporarily use Lagrangian coordinates which move with the fluid. Then $\mathbf{X}(x;t)$ (with x constant) is the trajectory of a fluid element with index x. $\mathbf{U}(x;t) \equiv \partial_t \mathbf{X}(x;t)$ is the velocity field of the membrane, and is a scalar under transformations of the internal coordinates. The geometrical change is given by the component of $\mathbf{U}$ normal to the surface $v_\perp \equiv \mathbf{N}\cdot\mathbf{U}$, while the tangential parts $u_i \equiv \partial_i \mathbf{X}\cdot\mathbf{U}$ describe the flow of material

$$\partial_t \mathbf{X} = v_\perp \mathbf{N} + u^i \partial_i \mathbf{X} \ .$$

Here $\mathbf{N}$ is the unit normal vector to the surface, and $\partial_i \mathbf{X}$ is a tangential vector with $\partial_i \equiv \partial/\partial x^i$. The first term describes the geometry, and the second term describes the flow.

If the density of the membrane constituents remains constant, the number of particles dN in a fluid element is proportional to $dA(x;t) \equiv \sqrt{g}(x;t) d^D x$. This yields the constraint $\partial_t \sqrt{g}(x;t) = 0$, where g is the determinant of the metric tensor $g_{ij} = \partial_i \mathbf{X}\cdot\partial_j \mathbf{X}$.

With the decomposition of $\partial_t \mathbf{X}$ in its tangential and normal part, and the representation of the extrinsic curvature tensor $K_{ij} = -\partial_i \mathbf{X}\cdot\partial_j \mathbf{N}$, one finds (see also [8])

$$\begin{aligned} 0 &= \partial_t \sqrt{\det(\partial_i \mathbf{X}\cdot\partial_j \mathbf{X})} \\ &= \sqrt{g}\, g^{ij} \partial_i \mathbf{X}\cdot\partial_j \partial_t \mathbf{X} \\ &= \sqrt{g}\, g^{ij} \partial_i \mathbf{X}\cdot\partial_j (v_\perp \mathbf{N} + u^k \partial_k \mathbf{X}) \\ &= \sqrt{g}\,(-K_i^i v_\perp + D_i u^i) \ . \end{aligned}$$

The local conservation of area can be written in manifestly covariant form:

$$v_\perp K = D_i u^i \ , \tag{2}$$

where K is the trace of K_{ij}, i.e., $K \equiv K_i^i = g^{ij} K_{ij}$, and D_i is the covariant derivative (for a short survey of differential geometry of manifolds, see [1]).

The fundamental degrees of freedom are the positions of the fluid elements $\mathbf{X}(x;t)$. For their motion we postulate a purely relaxational model obeying a detailed balance with the static model of David [9]. As already mentioned, the assumption of pure relaxation means that we are in the viscosity-dominated regime, corresponding to the Stokes regime in ordinary hydrodynamics. The constraint of locally constant area is enforced by a Lagrange multiplier $\phi(x;t)$.

In Lagrangian coordinates the model is

$$\partial_t \mathbf{X}(x;t) = -\lambda \frac{1}{\sqrt{g}(x)} \frac{\delta H}{\delta \mathbf{X}(x;t)} + \zeta(x;t) - \frac{1}{\sqrt{g}(x)} \int d^D y \, \phi(y) \frac{\delta \sqrt{g}(y)}{\delta \mathbf{X}(x;t)} \ , \tag{3}$$

with the constraint

$$\partial_t \sqrt{g}(x;t) = 0 \tag{4}$$

The factors $1/\sqrt{g}$ are needed to render the equation of motion covariant. A random force ζ has been included, but may be ignored if one is not interested in fluctuations of the membrane. The remaining deterministic part of the equation describes the macroscopic, overdamped motion of an incompressible membrane.

$H = H[\mathbf{X}]$ is the Helfrich Hamiltonian

$$H = \tfrac{1}{2} \int d^D x \sqrt{g} \, (\Delta_{\text{cov}} \mathbf{X})^2 = \int d^D x \sqrt{g} K^2/2 \ , \tag{5}$$

where Δ_{cov} is the covariant Laplacian (Laplace-Beltrami operator), and $K/2$ the mean curvature. Commonly a term proportional to the Gaussian curvature is added to H. The integral over the Gaussian curvature is a topological invariant, and unimportant if changes in the membrane topology do not occur. Next we are looking for a manifestly covariant representation of (3). First we calculate

$$\begin{aligned} \frac{1}{\sqrt{g}(x)} \int d^D y \, \phi(y) \frac{\delta \sqrt{g}(y)}{\delta \mathbf{X}(x)} &= \frac{1}{\sqrt{g}(x)} \int d^D y \, \phi(y) \sqrt{g}(y) g^{ij}(y) \partial_i \mathbf{X}(y) \frac{\partial}{\partial y^j} \delta(y-x) \\ &= -\frac{1}{\sqrt{g}} \partial_i (\phi \sqrt{g} g^{ij} \partial_j \mathbf{X}) \\ &= -g^{ij} \partial_i \phi \partial_j \mathbf{X} - \phi \Delta_{\text{cov}} \mathbf{X} \ . \end{aligned}$$

Together with the decomposition of $\partial_t \mathbf{X}$ and Eq. (2), we obtain

$$v_\perp \mathbf{N} + u^i \partial_i \mathbf{X} = -\lambda \frac{\delta_c H}{\delta_c \mathbf{X}} + g^{ij} \partial_i \phi \partial_j \mathbf{X} + \phi \Delta_{\text{cov}} \mathbf{X} \ ,$$

$$v_\perp K = D_i u^i \ ,$$

where $\delta_c H/\delta_c \mathbf{X} = (1/\sqrt{g}) \delta H/\delta \mathbf{X}$ is the covariant derivative of H with respect to $\mathbf{X}$. It can be shown [10] that

$$\frac{\delta_c H}{\delta_c \mathbf{X}} = (\Delta_{\text{cov}} K - \tfrac{1}{2} K^3 + K K^{ij} K_{ij}) \mathbf{N} \ , \tag{6}$$

i.e., this variational derivative is perpendicular to the surface. Finally, for the normal part of Eq. (3) we obtain

$$v_\perp = -\lambda \mathbf{N} \cdot \frac{\delta_c H}{\delta_c \mathbf{X}} + \phi K \ , \tag{7}$$

and, for the tangential part,

$$u_i = \partial_i \phi \ . \tag{8}$$

The Lagrangian multiplier ϕ can now be identified both as a varying surface tension and as the potential of the tangential velocity field. Substituting $u_i = \partial_i \phi$ into Eq. (2), we obtain the manifestly covariant equation

$$\partial_t \mathbf{X} \cdot \Delta_{\text{cov}} \mathbf{X} = v_\perp K = g^{ij} D_i \partial_j \phi = \Delta_{\text{cov}} \phi \ , \tag{9}$$

which determines ϕ up to a constant. Together with (7) this equation is the starting point of the following discussion. Occasionally we will compare this model with a simpler one without local area conservation, which has the equation of motion [11]

$$v_\perp = -\lambda \mathbf{N} \cdot \frac{\delta_c H}{\delta_c \mathbf{X}} + \lambda \sigma K \ , \tag{10}$$

with a constant surface tension σ instead of ϕ.

It is possible to eliminate the multiplier ϕ from Eqs. (7) and (9). This induces effective long-range forces. Multiplying Eq. (7) by K and substituting the result into Eq. (9) leads to

$$\Delta_{\text{cov}} \phi - K^2 \phi = -\lambda K \mathbf{N} \cdot \frac{\delta_c H}{\delta_c \mathbf{X}} \ , \tag{11}$$

and finally to

$$v_\perp = [I - K(-\Delta_{\text{cov}} + K^2)^{-1} K] \left[-\lambda \mathbf{N} \cdot \frac{\delta_c H}{\delta_c \mathbf{X}} \right] . \tag{12}$$

Here, I is the identity and $(-\Delta_{\text{cov}} + K^2)^{-1}$ the inverse of the operator $-\Delta_{\text{cov}} + K^2$. In the following sections we will apply the model to the relaxation of small deformations of cylindrical and planar stationary membrane configurations.

II. THE NEARLY CYLINDRICAL MEMBRANE

We first study the deterministic relaxation of perturbations of a cylindrical membrane, and calculate the dispersion rates of the elementary modes (for related problems, see also [5]). The cylinder is considered to be of fixed length (unlike in Ref. [12], where a cylinder of finite but variable length is considered), and we are interested in the limit of infinite length. The incompressibility of the membrane leads to an additional band of soft modes when there is no pressure difference between the inner and outer regions of the cylinders. Finally, we discuss the stability of the cylinder as a function of the pressure difference. For that purpose we add a force $\Delta p\mathbf{N}$ to our model, with the constant pressure different Δp. Then the equations of motion are

$$v_\perp = -\lambda(\Delta_{\rm cov}K - 1/2K^3 + KK^{ij}K_{ij}) + \phi K + \lambda\sigma K + \lambda\Delta p \ , \tag{13}$$

$$v_\perp K = \Delta_{\rm cov}\phi \ , \tag{14}$$

where $\lambda\sigma$ is the spatially constant part of the potential ϕ.

The perturbated surface is described by

$$\mathbf{X}(x,t) = \mathring{\mathbf{X}}(x) + \xi(x,t)\mathring{\mathbf{N}}(x) \ , \tag{15}$$

where ξ is the displacement normal to the reference membrane $\mathring{\mathbf{X}}$, and is a scalar field with respect to the metric induced by $\mathring{\mathbf{X}}$. Linearizing the equations in ξ and ϕ, we have $v_\perp = \partial_t\xi + O(\xi^2)$.

A stationary state of the membrane means $v_\perp = 0$ and $\phi = 0$. If the mean curvature $\mathring{K}$ is constant, this leads to the static equilibrium condition [10]

$$-\tfrac{1}{2}\mathring{K}^3 + \mathring{K}\mathring{K}_{ij}\mathring{K}^{ij} = \sigma\mathring{K} + \Delta p \ . \tag{16}$$

The cylinder of unit radius is represented by

$$\mathring{\mathbf{X}}(\psi,z) = \begin{pmatrix} \cos\psi \\ \sin\psi \\ z \end{pmatrix} \ . \tag{17}$$

With Eq. (16) we obtain the (formal) surface tension $\sigma = \frac{1}{2} + \Delta p$. The linearized equations of motion for the normal displacement are

$$\partial_t\xi = -\lambda[(\partial_\psi^2 + \partial_z^2)^2\xi + 2\partial_\psi^2\xi + \xi] - \phi + \lambda\Delta p[(\partial_\psi^2 + \partial_z^2)\xi + \xi] \ , \tag{18}$$

$$-\partial_t\xi = (\partial_\psi^2 + \partial_z^2)\phi \ , \tag{19}$$

which are solved with the ansatz

$$\xi(\psi,z;t) = \xi_0 e^{-\alpha t} e^{in\psi} e^{iqz} \ , \tag{20}$$

$$\phi(\psi,z;t) = \phi_0 e^{-\alpha t} e^{in\psi} e^{iqz} \ , \tag{21}$$

$$n = 0, \pm 1, \pm 2, \ldots ; -\infty < q < \infty \ .$$

This leads to

$$-\alpha\xi_0 = -\lambda[(n^2+q^2)^2 - 2n^2 + 1]\xi_0 - \phi_0 + \lambda\Delta p[-(n^2+q^2)+1]\xi_0 \ ,$$

$$\alpha\xi_0 = -(n^2+q^2)\phi_0 \ ,$$

and finally to the dispersion relation

$$\frac{\alpha}{\lambda} = \frac{n^2+q^2}{1+n^2+q^2}[(n^2+q^2)^2 - 2n^2 + 1 + \Delta p(n^2+q^2) - \Delta p] \ . \tag{22}$$

Without a pressure difference and without local area conservation (10), Eq. (22) reduces to

$$\frac{\alpha}{\lambda} = [(n^2+q^2)^2 - 2n^2 + 1] \ , \tag{23}$$

where, for stability reasons, the surface tension is set equal to $\sigma = \frac{1}{2}$. Table I shows the rates of dispersion to the lowest order in q for $\Delta p = 0$. In the case of an incompressible membrane the peristaltic modes ($n = 0$) form a band of soft modes. For these modes to relax, material has to flow over large distances which diverge in the limit $q \to 0$. The flow of material over a long distance requires a correspondingly long time. If there is no local area conservation, membrane material can appear and vanish instantaneously and thus the dispersion rate has a gap at $q = 0$. For both models bending modes (kink modes) form a soft band. However, the values of the relaxation rates in the limit $q \to 0$ for the model with incompressibility is half that of the model without local conservation. For higher bands, the differences between the two models decreases.

Critical pressure

With a nonzero pressure difference Δp we find the following dispersion rates for our model:

$$\frac{\alpha}{\lambda} = \frac{n^2+q^2}{1+n^2+q^2}[(n^2-1)^2 + 2n^2q^2 + q^4 + \Delta p(n^2+q^2-1)] \ . \tag{24}$$

For the ($n = 0$) band especially, we have

$$\frac{\alpha}{\lambda} = \frac{q^2}{1+q^2}[q^4 + 1 + \Delta p q^2 - \Delta p] = q^2(1-\Delta p) + O(q^4) \ , \tag{25}$$

which shows that the cylindrical configuration becomes unstable if $\Delta p > 1$. At $\Delta p = -2(1+\sqrt{2})$, the mode with

TABLE I. Rates of dispersion for the cylinder to the lowest order in q.

n	No local area— conservation $\alpha/\lambda=$	Local area— conservation $\alpha/\lambda=$
0 Peristaltic modes	$1+\cdots$	$q^2+\cdots$
± 1 Kink modes	$2q^2+\cdots$	$q^2+\cdots$
± 2	$9+\cdots$	$\frac{4}{5}\times 9+\cdots$

the longitudinal frequency $q^2=1+\sqrt{2}$ is marginal ($\alpha=0$). If the pressure difference Δp lies in the interval $-2(1+\sqrt{2})<\Delta p<0$, the dispersion relation is similar to that of rotons in superfluid ^{4}He.

For the $n=\pm 1$ band, we have

$$\frac{\alpha}{\lambda}=\frac{1+q^2}{2+q^2}[q^4+(2+\Delta p)q^2]$$
$$=\tfrac{1}{2}q^2(2+\Delta p)+O(q^4)\ , \tag{26}$$

so that these modes become unstable for $\Delta p<-2$.

Higher bands ($|n|>1$) are unstable for $\Delta p<1-n^2$ and otherwise stable. The relaxation rates are

$$\frac{\alpha}{\lambda}=\frac{n^2(n^2-1)}{1+n^2}[n^2-1+\Delta p]+O(q^2)\ .$$

Summarizing the linear stability analysis, we find that the infinitely long cylinder is stable against small deformations in the pressure difference interval $-2<\Delta p<1$, and unstable outside this interval. A negative Δp is equivalent to an excess pressure in the outer region.

III. RELAXATION OF A NEARLY PLANAR MEMBRANE

Since the mean curvature of a flat manifold vanishes, the condition of local area conservation introduces nonlinearities. These already show up in the relaxation of a nearly flat membrane in the lowest order of perturbation theory. In fact this leads to a nonlinear propagator of mean-field type, which is then used to calculate the dispersion of a Gaussian deformation of a planar membrane.

A nearly flat membrane in the Monge representation (see [13] and [1]) is described by

$$\mathbf{X}(\vec{x},t)=(\vec{x},f(\vec{x},t))\ ,$$

where $f(\vec{x},t)$ is the height of the surface at the point $\vec{x}$, which is a D dimensional Euclidean vector. It can be shown that $v_\perp=\dot{f}/\sqrt{g}$, so that our equations of motion (7) and (9) read

$$\frac{\dot{f}}{\sqrt{g}}=-\lambda\mathbf{N}\cdot\frac{\delta_c H}{\delta_c \mathbf{X}}+\phi(\vec{x})K+\lambda\sigma[f]K\ , \tag{27}$$

$$\frac{\dot{f}}{\sqrt{g}}K=\Delta_{\rm cov}\phi\ , \tag{28}$$

where

$$K=\partial_i\left[\frac{1}{\sqrt{g}}\partial_i f\right]$$

and $\sqrt{g}=\sqrt{1+\partial_i f\partial_i f}$. The zero mode (constant part) of ϕ is explictly written as $\lambda\sigma[f]$.

In an expansion around f we have to give a special value to σ in order to avoid infrared divergences. To the lowest order, we find

$$\dot{f}_0=-\lambda(\Delta^2 f_0-\sigma_0\Delta f_0)\ , \tag{29}$$

$$\Delta\phi_0=-\lambda[\Delta f_0(\Delta^2 f_0-\sigma_0\Delta f_0)]\ . \tag{30}$$

Hence ϕ_0 is well defined only if the spatial average of $\Delta f_0(\Delta^2 f_0-\sigma_0\Delta f_0)$ vanishes. This yields

$$\sigma_0=\frac{\int d^Dx\,\Delta f_0\Delta^2 f_0}{\int d^Dx(\Delta f_0)^2}\ . \tag{31}$$

The quantity σ_0 is a formal surface tension, and causes conservation of the total area $\int d^Dx\sqrt{g}$ to the lowest order:

$$\partial_t\int d^Dx\sqrt{g}=\int d^Dx\,\partial_t(\nabla f)\Big/2+O(f^4)$$
$$=\int d^Dx\,\Delta f\dot{f}+O(f^4)$$
$$=-\lambda\int d^Dx\,\Delta f(\Delta^2 f-\sigma\Delta f)+O(f^4)$$
$$=0+O(f^4)\ .$$

The physical reason for this is that modes with $p^2<|\sigma_0|$ are growing, and modes with $p^2>|\sigma_0|$ are decaying. Moreover, σ_0 allows an iteration of the perturbation expansion in f. Thus it is possible to determine ϕ to the lowest order of perturbation theory, which in turn can be used to obtain the next order in f.

IV. RELAXATION OF A LOCALIZED DEFORMATION

The deformation of a flat membrane at an arbitrary interior point is connected with a flow of material from infinite distances, because a deformed membrane has more area than an unperturbed flat one. Consequently, a localized perturbation will disperse in time, and the excess area will run off to infinity. We now will calculate the relaxation of a Gaussian deformation $f(\vec{x})\propto\exp[-\frac{1}{4}(\mu\vec{x})^2]$ to the lowest order, and determine the asymptotic time dependence of their typical length scale. In Fourier space, the equations of motion (29) read

$$\partial_t\tilde{f}(\vec{k},t)=-\lambda(k^4+\sigma k^2)\tilde{f}(\vec{k},t)\ ,$$

$$\sigma=-\frac{\int d^Dp\,p^6|\tilde{f}|^2(\vec{p})}{\int d^Dp\,p^4|\tilde{f}|^2(\vec{p})}\ ,$$

$$f_0(\vec{x},t)=\frac{1}{(2\pi)^D}\int d^Dp\,\tilde{f}(\vec{p},t)e^{i\vec{p}\cdot\vec{x}}\ .$$

The initial configuration is $\tilde{f}(k,t=0)=\tilde{f}_0\exp(-k^2/\mu^2)$. It can be seen that the integral equation for $\tilde{f}$ is

$$\tilde{f}(k,t)=\tilde{f}_0\exp\left[-\lambda k^4 t-\lambda k^2\int_0^t d\tau\,\sigma(\tau)-k^2\Big/\mu^2\right]\ . \tag{32}$$

In order to solve Eq. (32) it is convenient to introduce the moments

$$M_n(t):=\int_{k=0}^{\infty}dk\,k^{2n+D-1}\exp\left[-2\lambda k^4 t-2\lambda k^2\int_0^t d\tau\,\sigma(\tau)-2k^2\Big/\mu^2\right]\ , \tag{33}$$

where $k^{D-1}dk$ comes from the area element in the case of no angular dependence. In terms of these quantities we have

$$\sigma = -\frac{M_3}{M_2}$$

and

$$M_1 \propto \int d^Dx(\nabla f)^2 = 2\int d^Dx(\sqrt{g}-1)+O(f^4) .$$

A partial integration leads to the recursion relation

$$M_n = \frac{8\lambda t}{2n+D}M_{n+2} + \frac{4\lambda\int_0^t d\tau\,\sigma(\tau)}{2n+D}M_{n+1} + \frac{4/\mu^2}{2n+D}M_{n+1} .$$

From this equation we obtain for the ratios $Q_n := M_{n+1}/M_n$ the continued fractions

$$Q_n(t) = \frac{2n+D}{4\lambda\int_0^t d\tau\,\sigma(\tau) + \frac{4}{\mu^2} + 8\lambda t Q_{n+1}(t)} \tag{34}$$

and $\sigma(\tau) = -Q_2(\tau)$. The time derivative of Eq. (34) for $n=1$ leads to

$$\frac{d}{dt}\frac{1}{Q_0} = \frac{1}{D}(-4\lambda Q_2 + 8\lambda Q_1 + 8\lambda t\dot{Q}_1) . \tag{35}$$

On the other hand, the derivative of M_n with respect to time yields

$$\dot{M}_n = -2\lambda M_{n+2} + 2\lambda\frac{M_3}{M_2}M_{n+1} ,$$

which especially implies $\dot{M}_1 = -2\lambda M_3 + 2\lambda M_3 = 0$ and

$$\frac{d}{dt}\frac{1}{Q_0} = \frac{d}{dt}\frac{M_0}{M_1} = \frac{\dot{M}_0}{M_1} = -2\lambda\frac{M_2}{M_1} + 2\lambda\frac{M_3}{M_2}\frac{M_1}{M_1} = -2\lambda Q_1 + 2\lambda Q_2 . \tag{36}$$

If we eliminate $d/dt(1/Q_0)$ from the last two equations, we obtain (for $D=2$)

$$Q_2 = \tfrac{3}{2}Q_1 + t\dot{Q}_1 , \tag{37}$$

$$\frac{1}{Q_1} = -\lambda\int_0^t d\tau\,Q_2(\tau) + \frac{1}{\mu^2} + 2\lambda t Q_2 . \tag{38}$$

The second relation is Eq. (34) for $n=1$. Now we differentiate (38) with respect to time and substitute it back into (37). From this we obtain a nonlinear second order differential equation for Q_1:

$$2t^2\ddot{Q}_1 + \left[6t + \frac{1}{\lambda Q_1^2}\right]\dot{Q}_1 + \tfrac{3}{2}Q_1 = 0 , \tag{39}$$

which allows us to calculate the asymptotic time behavior of Q_1. For that purpose we introduce the dimensionless quantities A and B:

$$A(t) := t\frac{\dot{Q}_1}{Q_1} ,$$

$$B(t) := \frac{1}{\lambda t Q_1^2} ,$$

which obey the equations of motion

$$\frac{dA}{ds} = -2A - A^2 - \tfrac{3}{4} - \tfrac{1}{2}AB , \tag{40}$$

$$\frac{dB}{ds} = -B - 2AB , \tag{41}$$

where $d/ds := t\,d/dt$. This system of differential equations has the unstable fixed point (where $dA/ds = dB/ds = 0$) $(A^*,B^*) = (-\tfrac{3}{2},0)$ and the stable fixed point $(A^*,B^*) = (-\tfrac{1}{2},0)$. With our initial condition (A,B) runs into the stable fixed point. To study the behavior near the stable fixed point, we define the shifted variables $a := A+\tfrac{1}{2}, b := B$. The equations of motion now read

$$\frac{da}{ds} = -a^2 - a + \tfrac{1}{4}b - \tfrac{1}{2}ab , \tag{42}$$

$$\frac{db}{ds} = -2ab . \tag{43}$$

If a and b are small, a will quickly relax to $a_0(s) = 1/4b(s) + O(b^2)$. Due to this, b approximately obeys the equation

$$\frac{db}{ds} = -2a_c b = -\tfrac{1}{2}b^2 , \tag{44}$$

which is solved by $b(s) = b(\ln(t/t_0)) = 2/s$. Thus $Q_1 \sim t^{-1/2}\ln(t/t_0)^{1/2}$, $t\to\infty$, which implies

$$L \sim t^{1/4}\ln(t/t_0)^{-1/4} , \quad t\to\infty \tag{45}$$

for the characteristic length $L(t)$ of the deformation, since dimensionally $Q_1 \sim L^{-2}$. Obviously the equation of motion for a fluid membrane without conservation of area (10) leads to $L \sim t^{1/4}$. We see that incompressibility causes a logarithmically retarded dispersion of a localized perturbation of a flat membrane.

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$$D_t\rho = \partial_t\rho - g^{ij}\partial_t \mathbf{X}\cdot\partial_i \mathbf{X}\partial_j\rho$$

has to be used, where $\rho(x;t)$ is a scalar field, and g^{ij} is the inverse metric.

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